Proof-reading

Check that whether the following method is correct or not.

Given $\quad f(x)=\sqrt{\frac{x}{1+x^2}}\quad,\ x\geq 0\quad$, find its

(a) optimum point(s) ;

(b) point of inflection.

(a) Let $\quad f(x)=\sqrt{g(x)},\ g(x)=\frac{x}{1+x^2},\ x\geq 0$

$$g'(x)=\frac{(1+x^2)(1)-x(2x)}{(1+x^2)^2}=\frac{1-x^2}{(1+x^2)^2}=0\Rightarrow x=1\quad (x=-1\ \text{ is rejected})$$

When $0<x<1,\quad g'(x)<0\quad$ and $\quad$ when $\quad x>1,\quad g'(x)>0.$

By the First Derivative Test, $g(x)$ is a local max. when $x=1$.

Local Max. of $g(x)=\frac{1}{1+1^2}=\frac{1}{2}$.

Since $\quad \lim_{x\to+\infty} f(x)=\lim_{x\to+\infty}\sqrt{\frac{x}{1+x^2}}=\lim_{x\to+\infty}\sqrt{\frac{1}{\frac{1}{x}+x}}=0\quad$ and $\quad f(0)=0$.

Hence, $\quad$ the absolute maximum of $\quad f(x)=\sqrt{g(x)}=\sqrt{\frac{1}{2}}\approx \mathbf{0.707107}$ when $x=1$.

$\therefore\ f(0)=0$ is the absolute minimum.

(b) $$g'(x)=\frac{1-x^2}{(1+x^2)^2}=\frac{2-(1+x^2)}{(1+x^2)^2}=\frac{2}{(1+x^2)^2}-\frac{1}{1+x^2}$$

$$g''(x)=-\frac{8x}{(1+x^2)^3}+\frac{2x}{(1+x^2)^2}==\frac{-8x+2x(1+x^2)}{(1+x^2)^3}=\frac{2x(x^2-3)}{(1+x^2)^3}=0$$

$\therefore\ x=0,\pm\sqrt{3}$

Since $x\geq 0$, and obviously when $x=0$, $g(x)$ is not a point of inflection, $\therefore\ x=\sqrt{3}$.

Also $g''(x)$ changes sign when x goes through $\sqrt{3}$.

Therefore when $g(\sqrt{3})$ is a point of inflection.

$$f(\sqrt{3})=\sqrt{g(\sqrt{3})}=\sqrt{\frac{\sqrt{3}}{1+(\sqrt{3})^2}}=\frac{\sqrt[4]{3}}{2}$$

$\therefore$ Point of inflection is $\left(\boldsymbol{\sqrt{3},\frac{\sqrt[4]{3}}{2}}\right)$.

(a) Part (a) is still correct.

$$f(x)=\sqrt{\frac{x}{1+x^2}}\ ,\qquad f'(x)=\frac{1}{2}\sqrt{\frac{1+x^2}{x}}\frac{d}{dx}\left(\frac{x}{1+x^2}\right)=\frac{1}{2}\sqrt{\frac{1+x^2}{x}}\frac{(1+x^2)(1)-x(2x)}{(1+x^2)^2}=\frac{1}{2}\sqrt{\frac{1+x^2}{x}}\frac{1-x^2}{(1+x^2)^2}=0$$

$$\Longrightarrow x=1\quad (x=-1 \text{ is rejected})$$

When $0<x<1$, $f'(x)<0$ and when $x>1$, $f'(x)>0$.

By the First Derivative Test, $f(x)$ is a local max. when $x=1$.

Max. of $f(x)=\sqrt{\frac{1}{1+1^2}}=\sqrt{\frac{\mathbf{1}}{\mathbf{2}}}\approx \mathbf{0.707107}$.

(b) Part (b) is **NOT** correct. The differentiation is a bit longer, have patenice.

$$f'(x)=\frac{1}{2}\sqrt{\frac{1+x^2}{x}}\frac{1-x^2}{(1+x^2)^2}=\frac{1}{2}\left(\frac{1-x^2}{x^{\frac{1}{2}}(1+x^2)^{\frac{3}{2}}}\right)$$

Put $u=\frac{1}{2}\left(\frac{1-x^2}{x^{\frac{1}{2}}(1+x^2)^{\frac{3}{2}}}\right)$, $u^2=\frac{1}{4}\left(\frac{(1-x^2)^2}{x(1+x^2)^3}\right)$

$$2u\frac{du}{dx}=\frac{x(1+x^2)^3\frac{d}{dx}(1-x^2)^2-(1-x^2)^2\frac{d}{dx}x(1+x^2)^3}{4x^2(1+x^2)^6}=\frac{x(1+x^2)^3 2(1-x^2)(-2x)-(1-x^2)^2\left[(1+x^2)^3+3x(1+x^2)^2(2x)\right]}{4x^2(1+x^2)^6}$$

$$2f'(x)f''(x)=\frac{x(1+x^2)2(1-x^2)(-2x)-(1-x^2)^2[(1+x^2)+3x(2x)]}{4x^2(1+x^2)^4}=\frac{(1-x^2)\{-4x^2(1+x^2)-(1-x^2)[(1+x^2)+3x(2x)]\}}{4x^2(1+x^2)^4}$$

$$=\frac{(1-x^2)(3x^4-10x^2-1)}{4x^2(1+x^2)^4}$$

$$f''(x)=\frac{(1-x^2)(3x^4-10x^2-1)}{4x^2(1+x^2)^4}\times\frac{x^{\frac{1}{2}}(1+x^2)^{\frac{3}{2}}}{1-x^2}=\frac{\mathbf{3x^4-10x^2-1}}{\mathbf{4x^{\frac{3}{2}}(1+x^2)^{\frac{5}{2}}}}$$

$$f''(x)=0\Longrightarrow 3x^4-10x^2-1=0\quad\Longrightarrow x^2=\frac{5+2\sqrt{7}}{2}\quad (x>0)\quad\Longrightarrow x=\sqrt{\frac{5+2\sqrt{7}}{2}}$$

When x is slightly smaller than $\sqrt{\frac{5+2\sqrt{7}}{2}}$, x^2 is slightly smaller than $\frac{5+2\sqrt{7}}{2}$,

$3x^4-10x^2-1$ is slightly smaller than 0.

When x is slightly bigger than $\sqrt{\frac{5+2\sqrt{7}}{2}}$, x^2 is slightly bigger than $\frac{5+2\sqrt{7}}{2}$,

$3x^4-10x^2-1$ is slightly bigger than 0.

$f''(x)$ change sign as x goes through $x=\sqrt{\frac{5+2\sqrt{7}}{2}}$. It is a point of inflection.

$$f\left(\sqrt{\frac{5+2\sqrt{7}}{2}}\right)=\frac{\sqrt[4]{15+6\sqrt{7}}}{\sqrt{2(4+\sqrt{7})}}\ ,\qquad \text{Point of inflection is }\left(\sqrt{\frac{5+2\sqrt{7}}{2}},\ \frac{\sqrt[4]{15+6\sqrt{7}}}{\sqrt{2(4+\sqrt{7})}}\right)$$

Taking square root cannot change the optimum points of a curve (if the curve is well defined) but may change the points of inflection of the curve.

Small exercise

Find the point of inflection of the curve:

(a) $\mathrm{y} = (\mathrm{x}-1)\mathrm{x}(\mathrm{x}+1)+2$, where $-1 < x < 1$.

(b) $\mathrm{y} = \sqrt{(\mathrm{x}-1)\mathrm{x}(\mathrm{x}+1)+2}$, where $-1 < x < 1$.

(a) $(0\,,2)$

(b) $(0.04211\,,1.39927)$

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